ERROR IN SIMPSON'S RULE

Think of Taylor Series for $x_0 \le x \le x_M$ $\quad x_M - x_0 = 2h$

$$f(x) = f(x_1) + (x - x_1) f'(x_1) + \frac{(x - x_1)^2}{2!} f''(x_1) + \dots$$

Integration of this

$$\int_{x_0}^{x_M} f(x)\,dx = 2h\left[f(x_1) + \frac{1}{3}\frac{h^2}{2!} f''(x_1) + \frac{1}{5}\frac{h^4}{4!} f^{(4)}(x_1) + \dots\right]$$

Now Return to Simpson's Rule

$$\int_{x_0}^{x_M} f(x)\,dx \simeq \frac{h}{3}\left(\underbrace{f(x_0)}_{f(x_1 - k)} + 4f(x_1) + \underbrace{f(x_2)}_{f(x_1 + h)}\right)$$

Express them as Taylor Series

Then the equation comes:

$$\int_{x_0}^{x_M} f(x)\,dx \simeq \frac{h}{3}\Big[\Big(f(x_1) - hf'(x_1) + \frac{h^2}{2!} f''(x_1) - \dots\Big) + 4f(x_1)$$

$$+ \Big(f(x_1) + hf'(x_1) + \frac{h^2}{2!} f''(x_1) + \dots\Big)\Big]$$

$$= 2h\left(f(x_1) + \frac{1}{3}\frac{h^2}{2!} f''(x_1) + \frac{1}{3}\frac{h^4}{4!} f^{(4)}(x_1) + \dots\right)$$

Taylor Ser. Int.

$$\int_{x_0}^{x_m} f(x)\,dx = 2h\left[f(x_1)+\frac{1}{3}\frac{h^2}{2!}f''(x_1)+\frac{1}{5}\frac{h^4}{4!}f^{(4)}(x_1)+\dots\right]$$

Difference

Simpson Equation:

$$\int_{x_0}^{x_m} f(x)\,dx \cong 2h\left[f(x_1)+\frac{1}{3}\frac{h^2}{2!}f''(x_1)+\frac{1}{3}\frac{h^4}{4!}f^{(4)}(x_1)+\dots\right]$$

$$2h\cdot\left(-\frac{2}{15}\right)\frac{h^4}{4!}f^{(4)}(x_1)+\dots$$

$$\underbrace{= -\frac{h^5}{90}f^{(4)}(x_1)+\dots}_{\text{Error in Simpson's Rule}}$$