SIMPSON'S RULE BY NEWTON POLYNOMIALS

Let's take Newton Polynomial for 3 points.
The other constraint we have to comply with is that the points have equal distance between them. We will call it as 'h'.

$$N_3(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) + \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0} \cdot (x-x_0)(x-x_1)$$

$$x_1 - x_0 = x_2 - x_1 = h$$

and

$$s = \frac{x-x_0}{h}$$

$$N_3(x) = f(x_0) + \{f(x_1)+f(x_0)\} \cdot s + \underbrace{\frac{\frac{f(x_2)-f(x_1)}{h} - \frac{f(x_1)-f(x_0)}{h}}{h} \cdot (x-x_0)(x-x_1)}_{\frac{f(x_2)-2f(x_1)+f(x_0)}{2\cancel{h^2}} \cdot \overbrace{\frac{(x-x_0)}{h}}^{s} \cdot \overbrace{\frac{x-x_1}{h}}^{s-1}}$$

$$\int_0^2 N_3(x)\,ds = h\Big|_0^2 f(x_0)s + \frac{\{f(x_1)-f(x_0)\}s^2}{2} + \{f(x_2)-2f(x_1)+f(x_0)\}\left(\frac{s^3}{6}-\frac{s^2}{4}\right)$$

$$= h\left[2f(x_0) + 2f(x_1) - 2f(x_0) + \frac{f(x_2)}{3} - \frac{2}{3}f(x_1) + \frac{f(x_0)}{3}\right]$$

$$= h \cdot \left(\frac{f(x_0)}{3} + \frac{4}{3}f(x_1) + \frac{f(x_2)}{3}\right) \Rightarrow \text{Simpson's Rule}$$